A NOTE ON THE PERFORMER'S CONCEPTION OF SPACE

Abstract

Williams (1976) has drawn attention to the role of the group $Z_2 \times Z_2 \times Z_2$ in the dancer's conception of space. This note continues the group-theoretical analysis a little further.

Section 1: Geometric Space Versus the Performer's Conception of Space.

In this note we consider a very limited part of the performer's conception of space. We consider only directions in space, radiating out from a single fixed point. (Thus we consider only space from the point of view of a stationary performer.) Geometric (mathematical) space recognises infinitely many such directions; an example of a direction is '17 degrees left of straight ahead, and inclined upwards at an angle of 23 degrees to the horizontal'. Furthermore, mathematically, no direction is preferred over any other. We write $\underline{S}$ for the set of geometric directions (S for 'space').

We proceed from this geometrical space to the performer's conception of space in four 'acts'. These acts are acts of choice; they are not dictated by the structure of geometrical space (though they will be consistent with this structure). The choices we make are to reflect the way that a performer conceives space, so they are consequences of human anatomy and psychology. If, for example, we were beings with three legs and a triangular ground-plan, our choices would be very different.

Already in our description of a direction we have referred to the three directions 'left', 'straight ahead' and 'up'. So, our first act is to impose three mutually orthogonal preferred directions.

Our second act recognizes that a performer cannot distinguish among infinitely many different directions; we consider only a finite set of directions. To be specific, we will consider 26 directions, as follows:

U (up)	D (down)
R (right)	L (left)
F (front)	B (back)

Fl (front left), FR, BL, BR, FU, FD, BU, BD, LU, LD, RU, RD
FLU (front left up), FLD, FRU, FRD, BLU, BLD, BRU, BRD.

We call this set of 26 directions D. The choice of these 26 directions is not absolute, but covers many practical situations.

Our third act recognises that a direction like RU need not be exactly 45 degrees between R and U. We regard the 26 directions listed above as abstract directions or as symbols for directions. (The second and third acts recognise the human predilection to treat similar things as identical.)

A realization of the abstract directions assigns to each abstract direction an actual direction in space. (So a realization is a function from D to S.) We insist that every realization assigns to the symbol U the direction straight up, to the symbol D the direction straight down, and similarly for F, R, B and L. However, we only ask that a realization assigns to RU a direction somewhere between right and up; one realization might assign to RU the direction inclined upwards 30 degrees from right, and another the direction inclined upwards 50 degrees from right.

So far we have expressed our version of the performer's conception of space as a set of 26 abstract directions, capable of many realizations in geometrical space. However, one more act remains.

Section 2: Introduction of a Symmetry Group.

A symmetry group is a collection of transformations that act on some set. (that is, each transformation is a function from the set to itself.) We consider transformations acting on our set D of 26 abstract directions. An example of a transformation is the operation that interchanges up and down. We will call this operation o_{UD}. The effect of o_{UD} on the abstract directions is as follows:

o_{UD}(U) = D	o_{UD}(D) = U
o_{UD}(R) = R	o_{UD}(L) = L
o_{UD}(F) = F	o_{UD}(B) = B
o_{UD}(FL) = FL	o_{UD}(FR) = FR
o_{UD}(BL) = BL	o_{UD}(BR) = BR
o_{UD}(FU) = FU	o_{UD}(FD) = FU
o_{UD}(BU) = BD	o_{UD}(BD) = BU
o_{UD}(LU) = LD	o_{UD}(LD) = LU
o_{UD}(RU) = RD	o_{UD}(RD) = RU
o_{UD}(FLU) =FLD	o_{UD}(FLD) =FLU
o_{UD}(FRU) =FRD	o_{UD}(FRD) =FRU
o_{UD}(BLU) =BLD	o_{UD}(BLD) =BLU
o_{UD}(BRU) =BRD	o_{UD}(BRD) =BRU

Two other transformations similar to o_{UD} are o_{LR}, which interchanges left and right, and o_{FB}, which interchanges front and back. Two transformations can be combined by performing first one and then the other. If we begin with o_{UD}, o_{LR} and o_{FB} and form all possible combinations we end up with eight transformations which together make up a symmetry group we call G_B. (This notation will be explained later.) The eight transformations in G_B are:

o_{UD}, o_{LR}, o_{FB}

the transformation that interchanges both U and D and L and R (leaving F and B unaltered)

the transformation that interchanges both U and D and F and B (leaving L and R unaltered)

the transformation that interchanges both L and R and F and B (leaving U and D unaltered)

the 'central symmetry' that interchanges each direction with its opposite

the identity transformation (which leaves all directions unaltered).

The identity transformation may seem superfluous, but it is needed to ensure that the combination of two transformations in G_B is always another transformation in G_B. This group is isomorphic to the group $Z_2 \times Z_2 \times Z_2$ (referred to in Williams [1976] as K x K x K). Our fourth act is to introduce the symmetry group G_B to our description.

The introduction of a symmetry group has two consequences. Firstly, it imposes a notion of 'relatedness' (or 'being of a similar kind') on the set of abstract directions. We naturally regard U and D as being more closely related than U and FR, say. The group G_B breaks up the set of 26 abstract directions into small sets called <u>orbits</u>; two abstract directions are in the same orbit if there is a transformation in G_B that takes one direction to the other. The orbits of D under G_B are:

{U,D} {R,L} {F,B}
{FL,FR,BL,BR} {FU,FD,BU,BD} {LU,LD,RU,RD}
{FLU,FLD,FRU,FRD,BLU,BLD,BRU,BRD}

We will regard two abstract directions as related if they are in the same orbit under G_B.

Secondly, the symmetry group affects the realizations of the abstract directions as actual directions. The transformation o_{UD}, acting on D, has a counterpart $_{UD}$ which acts on the set S of actual directions by interchanging up and down; o_{UD} is the reflection in the horizontal plane through the fixed point from which the directions radiate. Similarly, each of the other transformations in G_B has a counterpart which is a transformation of S. The set of all these counterparts forms a group we call H_B. H_B is the group of symmetries of a rectangular box (with depth, width and height all different); the subscript B in the notations G_B and H_B stands for 'box'. The eight transformations in H_B are (listed in the order corresponding to the list above of the transformations in G_B):

the reflection in the horizontal plane (Williams [1976] "transverse plane")

the reflection in the sagittal plane (dividing right from left)

the reflection in the coronal plane (dividing front from back)

rotation 180 degrees about the front-back axis

rotation 180 degrees left-right axis

rotation 180 degrees about the up-down axis

the reflection in the central point (interchanging each direction with its opposite)

the identity transformation (leaving all directions unaltered).

(H_B is isomorphic to G_B and to Z_2 x Z_2 x Z_2.)

We ask that 'realizations have the symmetry of the rectangular box'. Precisely: If r :D   S is a realization and d   D is an abstract direction, then $r(o_{UD}(d) =$ $_{UD}(r[d])$, and similarly for each of the other transformations in G_B and its counterpart in H_B. This means that if r(FU) (that is, the realization of FU) is inclined upwards at 30 degrees to the horizontal, then r(BU) must also be inclined upwards at 30 degrees and r(FD) and r(BD) must both be inclined downwards at 30 degrees. Roughly speaking, the angles of inclination of the realizations of abstract directions in the same orbit must be equal.

Our version of the performer's conception of space is now a set of 26 abstract directions acted on by the group Z_2 x Z_2 x Z_2.

This set of directions has many realizations in geometrical space, but we insist that any such realization has the symmetry of the rectangular box.

The description just given explicates the appearance of the group $Z_2 \times Z_2 \times Z_2$ in Williams (1976); this group appears as a group of transformations acting on both our set of 26 abstract directions, and on the infinite set of actual directions in space; the two actions are related through the allowable realizations.

Section 3: Refinements.

The analysis in the previous section accounts for what Williams says about $Z_2 \times Z_2 \times Z_2$. However, it is interesting to see if we can proceed further. We indicated that the orbits impose a notion of 'relatedness' on D such that, for example, LU and LD are related, but LU and FU are not, and neither are LU and FLU. However, we feel that LU and FU are somewhat related, more so than LU and FLU. We will proceed by considering other groups. We regard the analysis of section 2 as basic, so we insist that any group considered contains $Z_2 \times Z_2 \times Z_2$ (that is, contains the three fundamental transformations o_{UD}, o_{LR} and o_{FB}).

The choice of groups is very limited: in fact, there are only three (Coxeter [1969], Chapter 15, has a complete list of the finite symmetry groups). As transformations of S the three groups are:

the group of symmetries of a square prism;

the group of symmetries of an octagonal prism;

the group of symmetries of a cube.

We call these groups respectively H_{SP}, H_{OP} and H_C; the corresponding groups of symmetries of D we call G_{SP}, G_{OP} and G_C. (This continues the notation G_B and H_B used in section 2.) These groups are related as follows:

$$G_B$$

$$G_{SP}$$

$$G_{OP} \qquad\qquad G_C$$

Here the presence of a line joining two groups means that the upper group is contained in the lower.

Although there are only three available groups, in the case of G_{SP} and G_{OP} we do have a further choice, namely the direction of the main axis of symmetry. It is natural to choose this axis to be vertical, and we do so.

G_{SP} is a group containing 16 transformations. The orbits of <u>D</u> and G_{SP} are:

{U,D}
{F,L,B,R}
{FU,FR,BL,BR}
{FU,FD,LU,LD,RU,RD,BU,BD}
{FLU,FLD,FRU,FRD,BLU,BLD,BRU,BRD}

We could call two elements of <u>D</u> <u>strongly related</u> if they are in the same orbit under G_B, and <u>weakly related</u> if they are in different orbits under G_B, but in the same orbit under G_{SP}. Thus LU and LD are strongly related, LU and FU are weakly related, and LU and FLU are not related at all (neither strongly related nor weakly related). This seems in accord with our intuition. The group G_{SP} introduces rotations of 90 degrees in the horizontal plane.

Introducing one of the groups G_{OP} or G_C amounts to introducing a notion of 'very weakly related'. We consider G_{OP} first.

G_{OP} is a group containing 32 transformations. The orbits of <u>D</u> under G_{OP} are:

{U,D}
{F,FL,L,BL,B,BR,R,FR}
{FU,FD,LU,LD,RU,RD,BU,BD,FLU,FLD,FRU,FRD,BLU,BLD,BRU,BRD}

This seems a plausible notion of 'very weakly related'. The eight horizontal directions in <u>D</u> are all lumped together by G_{OP}; the group G_{OP} introduces rotations of 45 degrees in the horizontal plane.

G_C is a group containing 48 transformations. The orbits of <u>D</u> under G_C are:

{F,L,B,R,U,D}
{FU,FD,FL,FR,LU,LD,RU,RD,BU,BD,BL,BR}
{FLU,FLD,FRU,FRD,BLU,BLD,BRU,BRD}

At first sight this also looks promising as a notion of 'very weakly related' (evidently different from that associated with G_{OP}). However, G_C contains transformations that are definitely not part of the performer's conception of space. One of them (which I will call ' is the following:

(F) = U	(U) = L	(L) = F
(B) = D	(D) = R	(R) = B
(FU) = LU	(LU) = FL	(FL) = FU
(RD) = BR	(BR) = BD	(BD) = RD
(RU) = LB	(LB) = FD	(FD) = RU
(BU) = LU	(LD) = RF	(RF) = BU
(FLD) = FRU	(FRU) = BLU	(BLU) = FLD
(BRU) = BlD	(BLD) = FRD	(FRD) = BLU
(FLU) = FLU	(BRD) = BRD	

This transformation corresponds to the rotation of the cube about 120 degrees through the axis joining the corners FLU and BRD. It is a fact that if one holds a cube by two diagonally opposite corners and rotates it by 120 degrees, the cube will be superimposed on its original position. So a cube has <u>three-fold</u> symmetry about the axis joining two opposite corners. This three-fold symmetry is something of a surprise to many people, and doesn't correspond to anything in the performer's conception of space. For this reason we can exclude G_C from consideration.

This means that there is only one natural notion of 'very weakly related' on our set of abstract directions, namely, that determined by G_{OP}.

<u>Section 4: Conclusion</u>.

It is not reasonable to expect a complete mathematical description of the performer's conception of space. At most we can expect to model what are generally felt to be the most important features, and there may indeed be disagreement about what the most important features are.

The model presented here includes the following:

Only a finite number of directions are distinguished.

The approximate character of many of these directions is taken into account by the introduction of 'abstract directions'.

The basic role of the three fundamental oppositions up/down, left/right and front/back is recognised by requiring the three transformations o_{UD}, o_{LR} and o_{FB} to be included in the symmetry groups we consider.

The introduction of a symmetry group enables us to consider some abstract directions as more closely related than others.

The introduction of further groups in section 3 achieves two things: it introduces a hierarchy of degrees of relatedness on the abstract directions, and it is a step towards allowing for the fact that we may have different needs at different times: for example, if we need to consider exact 45 degrees rotations in the horizontal plane, we should work with G_{OP}. In other words, the perceived symmetry of the space may change; since a symmetry group is the mathematical expression of symmetry, our symmetry group should change accordingly.

The choice of 26 abstract directions in section 1 constrains the available symmetry groups to those considered in section 2 and section 3. A different choice of abstract directions would allow the use of different symmetry groups. However, human anatomy accounts for the importance of the oppositions up/down/ left/right and front/back, and makes the set of 26 abstract directions used in this paper a natural choice.

The starting point of this paper was as follows: The group $Z_2 \times Z_2 \times Z_2$ is clearly important in Williams (1976), but is not presented there as a group of transformations. However, a symmetry group should appear as a group of transformations; the question raised is 'transformations of what'? The answer given in this paper is that $Z_2 \times Z_2 \times Z_2$ in fact appears in two different guises as a transformation group: firstly, acting on the set $\underline{D}$ of abstract directions, and secondly, acting on the set $\underline{S}$ of geometric directions. (These two guises are called respectively G_B and H_B in section 2.) The link between the two guises of $Z_2 \times Z_2 \times Z_2$ is the requirement that realizations of abstract directions have the symmetry of the rectangular box.

It is not claimed that this paper represents the last word on the subject. For example, in section 3 we chose the axis of symmetry of G_{SP} to be vertical. The fact that this is a natural choice indicates that the three fundamental transformations o_{UD}, o_{LR}, o_{FB} are not all on the same footing; o_{UD} is perhaps less fundamental than the other two. We could then consider symmetry groups containing o_{LR} and o_{FB} but not o_{UD}. This would extend the hierarchy of symmetry groups used in the model.

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